IEOR 240 LP Theory Notes

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We shall consider in these notes the following LP in symmetric form:

$$
\begin{aligned}
\mathcal{P}: \quad & \min \boldsymbol{c}^\intercal \boldsymbol{x} \\
\text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \ge \boldsymbol{b} \qquad (1)\\
& \boldsymbol{x} \ge \boldsymbol{0} \qquad (2)
\end{aligned}
$$

where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$ and $\boldsymbol{c}, \boldsymbol{x} \in \mathbb{R}^n$.

Definitions:

- We say that $\bar{\boldsymbol{x}}$ is a *feasible solution* for $\mathcal{P}$ if $\bar{\boldsymbol{x}}$ satisfies (1) and (2).
- If $\mathcal{P}$ has a feasible solution then we say that $\mathcal{P}$ is *feasible*; otherwise we say that $\mathcal{P}$ is *infeasible*.
- $\mathcal{P}$ is said to be *unbounded* if for every scalar α there exists a feasible solution $\bar{\boldsymbol{x}}$ for $\mathcal{P}$ such that $\boldsymbol{c}^\intercal\bar{\boldsymbol{x}} \le \alpha$.
- Suppose $\mathcal{P}$ is feasible. Then, $\mathcal{P}$ is said to be *bounded* if there exists a scalar α such that for *every* feasible solution $\bar{\boldsymbol{x}}$ for $\mathcal{P}$, we have $\boldsymbol{c}^\intercal\bar{\boldsymbol{x}} \ge \alpha$.
- $\boldsymbol{x}^*$ is an *optimal solution* for $\mathcal{P}$ if $\boldsymbol{x}^*$ is a feasible solution for $\mathcal{P}$ and $\boldsymbol{c}^\intercal\boldsymbol{x}^* \le \boldsymbol{c}^\intercal\bar{\boldsymbol{x}}$ for *every* feasible solution $\bar{\boldsymbol{x}}$.

Our goal, which is the key to a useful mathematical theory of LP, is to develop necessary and sufficient conditions for all possible outcomes for $\mathcal{P}$, such as infeasibility, unboundedness, or optimality. In fact, it is possible to construct a comprehensive theory of such conditions by simple manipulations of the necessary and sufficient conditions for the feasibility of $\mathcal{P}$.

Obviously, any feasible solution certifies that $\mathcal{P}$ is feasible. Thus, we say that $\bar{\boldsymbol{x}}$ satisfying (1)-(2) is a *feasibility certificate* for $\mathcal{P}$. The following theorem provides easily verifiable sufficient condition for $\mathcal{P}$ to be infeasible.

Theorem 1 (Infeasibility Certificate) *If there exists* $\boldsymbol{z} \in \mathbb{R}^m$ *such that*

$$
\begin{aligned}
\boldsymbol{z}^\intercal\boldsymbol{A} &\le \boldsymbol{0}, \qquad (3)\\
\boldsymbol{z} &\ge \boldsymbol{0}, \qquad (4)\\
\boldsymbol{b}^\intercal\boldsymbol{z} &> \boldsymbol{0}, \qquad (5)
\end{aligned}
$$

then $\mathcal{P}$ is infeasible.

Proof. Suppose, to the contrary, that there exist $\bar{\boldsymbol{x}}$ satisfying (1) and (2), and $\bar{\boldsymbol{z}}$ satisfying (3)-(5). Then,

$$0 \geq (\bar{\boldsymbol{z}}^\mathsf{T}\boldsymbol{A})\bar{\boldsymbol{x}} = \bar{\boldsymbol{z}}^\mathsf{T}\boldsymbol{A}\bar{\boldsymbol{x}} = \bar{\boldsymbol{z}}^\mathsf{T}(\boldsymbol{A}\bar{\boldsymbol{x}}) \geq \bar{\boldsymbol{z}}^\mathsf{T}\boldsymbol{b} = \boldsymbol{b}^\mathsf{T}\boldsymbol{z} > 0,$$

leading to the conclusion that $0 > 0$, a contradiction. ■

If $\bar{\boldsymbol{z}}$ satisfies (3)-(5), we call it an *infeasibility certificate* for $\mathcal{P}$. While it is not obvious, it can be shown that whenever $\mathcal{P}$ is infeasible, there exists $\boldsymbol{z}$ satisfying (3)-(5). This result is presented (without a proof) in the following theorem, which is known as *Farkas' Lemma.*

Theorem 2 (Farkas' Lemma) *For any data*[1] *of $\mathcal{P}$ exactly one the following is true:*

- *There exists $\boldsymbol{x} \in \mathbb{R}^n$ satisfying (1) and (2).*
- *There exists $\boldsymbol{z} \in \mathbb{R}^m$ satisfying (3)-(5).*

Next, we construct a certificate for unboundedness of $\mathcal{P}$. The following theorem provides easily verifiable sufficient conditions for $\mathcal{P}$ to be unbounded if feasible.

Theorem 3 (Unboundedness Certificate) *If there exists $\boldsymbol{w} \in \mathbb{R}^n$ such that*

$$\boldsymbol{A}\boldsymbol{w} \geq \boldsymbol{0}, \tag{6}$$
$$\boldsymbol{w} \geq \boldsymbol{0}, \tag{7}$$
$$\boldsymbol{c}^\mathsf{T}\boldsymbol{w} > \boldsymbol{0}, \tag{8}$$

then $\mathcal{P}$ is unbounded if feasible.

Proof. Suppose $\bar{\boldsymbol{w}}$ satisfies (6)-(8), and let $\bar{\boldsymbol{x}}$ be a feasible solution for $\mathcal{P}$ (that is, $\bar{\boldsymbol{x}}$ satisfies (1) and (2)). Now, define $\boldsymbol{x}(\lambda) = \bar{\boldsymbol{x}} + \lambda\bar{\boldsymbol{w}}$. We claim that for all $\lambda \geq 0, \boldsymbol{x}(\lambda)$ is feasible for $\mathcal{P}$. The claim is verified by noting that for $\lambda \geq 0$,

$$\boldsymbol{A}\boldsymbol{x}(\lambda) = \boldsymbol{A}(\bar{\boldsymbol{x}} + \lambda\bar{\boldsymbol{w}}) = A\bar{\boldsymbol{x}} + \lambda\boldsymbol{A}\bar{\boldsymbol{w}} \geq \boldsymbol{b} + \lambda\boldsymbol{0} = \boldsymbol{b},$$

In addition, for $\lambda \geq 0$, $\boldsymbol{x}(\lambda) = \bar{\boldsymbol{x}} + \lambda\bar{\boldsymbol{w}} \geq \boldsymbol{0}$, so $\boldsymbol{x}(\lambda)$ satisfies (1) and (2).

However, since

$$\boldsymbol{c}^\mathsf{T}\boldsymbol{x}(\lambda) = \boldsymbol{A}(\bar{\boldsymbol{x}} + \lambda\bar{\boldsymbol{w}}) = \boldsymbol{c}^\mathsf{T}\bar{\boldsymbol{x}} + \lambda\boldsymbol{c}^\mathsf{T}\bar{\boldsymbol{w}} \ \ (\text{where } \boldsymbol{c}^\mathsf{T}\bar{\boldsymbol{w}} < \boldsymbol{0}),$$

$\mathcal{P}$ is clearly unbounded, as $\boldsymbol{x}(\lambda)$ is feasible for any $\lambda \geq 0$, so λ can be made arbitrarily large, maintaining feasibility and driving (since $\boldsymbol{c}^\mathsf{T}\bar{\boldsymbol{w}} < \boldsymbol{0}$) the objective function towards

[1] That is, $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}$.

$-\infty$. In particular, for any given α, we can find a non-negative λ for which $\boldsymbol{c}^\intercal \boldsymbol{x}(\lambda) \leq \alpha$ (in fact, $\lambda = \max\{0, \frac{\alpha - \boldsymbol{c}^\intercal \boldsymbol{x}(\lambda)}{\boldsymbol{c}^\intercal \bar{\boldsymbol{w}}}\}$ will do). ∎

We say that $\bar{\boldsymbol{w}}$ satisfying (6)-(8) is an *unboundedness certificate* for $\mathcal{P}$. Note that for $\mathcal{P}$ to be recognized as unbounded, we need both feasibility and unboundedness certificates.

Next, we introduce easily verifiable sufficient conditions for boundedness of $\mathcal{P}$ if feasible.

Theorem 4 (Boundedness Certificate) *Let $\boldsymbol{x}$ be a feasible solution for $\mathcal{P}$ (that is, $\boldsymbol{x}$ satisfies (1) and (2)), and suppose $\boldsymbol{y} \in \mathbb{R}^m$ satisfies*

$$\boldsymbol{y}^\intercal \boldsymbol{A} \leq \boldsymbol{c}^\intercal \quad (\boldsymbol{A}^\intercal \boldsymbol{y} \leq \boldsymbol{c}) \tag{9}$$

$$\boldsymbol{y} \geq \boldsymbol{0} \tag{10}$$

then $\boldsymbol{c}^\intercal \boldsymbol{x} \geq \boldsymbol{b}^\intercal \boldsymbol{y}$.

Proof. The proof follows easily by noticing that

$$\boldsymbol{c}^\intercal \boldsymbol{x} \geq (\boldsymbol{y}^\intercal \boldsymbol{A})\boldsymbol{x} = \bar{\boldsymbol{y}}^\intercal \boldsymbol{A} \bar{\boldsymbol{x}} = \boldsymbol{y}^\intercal (\boldsymbol{A}\boldsymbol{x}) \geq \boldsymbol{y}^\intercal \boldsymbol{b} = \boldsymbol{b}^\intercal \boldsymbol{y}.$$ ∎

This last theorem establishes that if $\bar{\boldsymbol{y}}$ satisfy (9) and (10), then $\boldsymbol{b}^\intercal \boldsymbol{y}$ is a lower bound for the objective function value of *every* feasible solution of $\mathcal{P}$. Thus, we say that $\bar{\boldsymbol{y}}$ satisfying (9) and (10) is a *boundedness certificate* for $\mathcal{P}$. Note that for $\mathcal{P}$ to be recognized as bounded, we need both feasibility and boundedness certificates.

Now that we have identified boundedness and unboundedness certificates, we need to find out whether an appropriate certificate always exists. In particular, we obviously have that if $\mathcal{P}$ is feasible than it must be either bounded or unbounded (but not both). So the question is whether it is guaranteed that whenever a feasible $\mathcal{P}$ is bounded (unbounded), a bounded (unbounded) certificate exists. In the following theorem we show (by applying Farkas' Lemma) that the answer is 'yes'!

Theorem 5 *For any data of $\mathcal{P}$ exactly one the following is true:*

- *There exist $\boldsymbol{y}$ satisfying (9)-(10).*
- *There exist $\boldsymbol{w}$ satisfying (6)-(8).*

Proof. Multiplying the inequalities in (9) by -1, we obtain the following system of inequalities, which is equivalent to (9)-(10):

$$-\boldsymbol{y}^\intercal \boldsymbol{A} \geq -\boldsymbol{c}^\intercal \tag{9'}$$

$$\boldsymbol{y} \geq \boldsymbol{0} \tag{10'}$$

Similarly, by multiplying the inequalities in (6) and inequality (8) by -1, we obtain the following system inequalities, which is equivalent to (6)-(8):

$$-\boldsymbol{A}\boldsymbol{w} \leq \mathbf{0} \tag{6'}$$
$$\boldsymbol{y} \geq \mathbf{0} \tag{7'}$$
$$-\boldsymbol{c}^{\mathsf{T}}\boldsymbol{w} < \mathbf{0} \tag{8'}$$

Noticing that (by renaming coefficients, variables, and summations indices) the correspondence between (9')-(10') and (6')-(8') is the same as the correspondence between (1)-(2) and (3)-(5), and invoking Farkas' Lemma (Theorem 2), complete the proof. ■

As an immediate corollary to the preceding theorem, it is clear that if we have a feasible solution to $\mathcal{P}$ whose objective function value is the same as a lower bound for $\mathcal{P}$, then this solution is optimal for $\mathcal{P}$. This observation leads to the following easily verifiable sufficient conditions for optimal solution for $\mathcal{P}$.

Theorem 6 (Optimality Certificate) *Let $\boldsymbol{x}$ be a feasible solution for $\mathcal{P}$, $\boldsymbol{y}$ satisfies* (9) *and* (10)*, and $\mathbf{c}^{\mathsf{T}}\boldsymbol{x} = \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y}$. Then, $\boldsymbol{x}$ is an optimal solution for $\mathcal{P}$.*

Proof. Follows immediately from Theorem 4. ■

Thus, given $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ satisfying the conditions in Theorem 6, we say that $\boldsymbol{y}^*$ is an *optimality certificate* for $\boldsymbol{x}^*$. In addition, it can be shown that whenever $\boldsymbol{x}^*$ is an optimal solution for $\mathcal{P}$, there exists $\boldsymbol{y}^*$ certifying its optimality. In fact, this result is usually presented in somewhat different form as follows (the proof, which is based on Farkas' Lemma is omitted).

Theorem 7 *Suppose $\mathcal{P}$ is feasible and bounded* (*that is, there exists $\bar{\boldsymbol{x}}$ satisfying* (1) *and* (2)*, and $\bar{\boldsymbol{y}}$ satisfying* (9) *and* (10))*, then there exist $\boldsymbol{x}^*$ satisfying* (1) *and* (2)*, and $\boldsymbol{y}^*$ satisfying* (9) *and* (10)*, such that $\mathbf{c}^{\mathsf{T}}\boldsymbol{x}^* = \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y}^*$.*

Note that Theorem 7 establishes that if $\mathcal{P}$ is feasible and bounded, then it must have an optimal solution.

Summarizing the discussion above we have:

- Either there exists $\bar{\boldsymbol{x}}$ satisfying (1) and (2) (in which case $\mathcal{P}$ is feasible), or there exists $\bar{\boldsymbol{z}}$ satisfying (3)-(5) (in which case $\mathcal{P}$ is infeasible).
- Either there exists $\bar{\boldsymbol{w}}$ satisfying (6)-(8) (in which case $\mathcal{P}$ is unbounded if feasible), or there exists $\bar{\boldsymbol{y}}$ satisfying (9) and (10) (in which case $\mathcal{P}$ is bounded if feasible).
- If $\mathcal{P}$ is both feasible and bounded (that is, there exist $\bar{\boldsymbol{x}}$ satisfying (1) and (2), and $\bar{\boldsymbol{y}}$ satisfying (9) and (10)), then there exists an optimal solution $\boldsymbol{x}^*$ to $\mathcal{P}$ and a corresponding optimality certificate $\boldsymbol{y}^*$.

From the preceding we can construct the following table, displaying the status of $\mathcal{P}$ for any of the four combinations of existence of the certificates discussed in the first two bullets above.

	There exists $\bar{\boldsymbol{x}}$ **satisfying** (1) **and** (2)	**There exists** $\bar{\boldsymbol{z}}$ **satisfying** (3)**-**(5)
There exists $\bar{\boldsymbol{y}}$ **satisfying** (9) **and** (10)	There exists an optimal solution $\boldsymbol{x}^*$ certified by some $\boldsymbol{y}^*$	$\mathcal{P}$ is infeasible
There exists $\bar{\boldsymbol{w}}$ **satisfying** (6)**-**(8)	$\mathcal{P}$ is unbounded	$\mathcal{P}$ is infeasible

It is useful to combine Theorems 6 and 7 to obtain necessary and sufficient conditions for optimality as follows:

Optimality Conditions I
$\boldsymbol{x}^*$ is optimal for $\mathcal{P}$ if and only if there exists $\boldsymbol{y}^*$ such that:

(a) $\boldsymbol{x}^*$ satisfies (1) and (2),

(b) $\boldsymbol{y}^*$ satisfies (9) and (10),

(c) $\boldsymbol{c}^\intercal \boldsymbol{x}^* = \boldsymbol{b}^\intercal \boldsymbol{y}^*$.

These conditions are often expressed as follows:

Optimality Conditions II
$\boldsymbol{x}^*$ is optimal for $\mathcal{P}$ if and only if there exists $\boldsymbol{y}^*$ such that:

(a') $\boldsymbol{x}^*$ satisfies (1) and (2),

(b') $\boldsymbol{y}^*$ satisfies (9) and (10),

(c') $\boldsymbol{r}_j^* \boldsymbol{x}_j^* = 0$ $(j = 1, \ldots, n)$, $\boldsymbol{s}_i^* \boldsymbol{y}_i^* = 0$ $(i = 1, \ldots, m)$.

(where $\boldsymbol{s}^* = \boldsymbol{A}\boldsymbol{x}^* - \boldsymbol{b}$ and $\boldsymbol{r}^* = \boldsymbol{c} - \boldsymbol{A}^\intercal \boldsymbol{y}^*$)

The conditions in (c') above are called *complementary slackness conditions* [2] (CSC).

[2] The name comes from the fact that the $\boldsymbol{s}_i^*$'s ($\boldsymbol{r}_j^*$'s) are the slacks of (1) ((9)) and that $\boldsymbol{x}_j^*$'s ($\boldsymbol{y}_i^*$'s) are the slacks of (2) ((10)).

Theorem 8 *Optimality conditions I are satisfied by $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ if and only if they satisfy optimality conditions II.*

Proof. Obviously, conditions (a) and (b) in optimality conditions I are identical to conditions (a') and (b') in optimality conditions II. Now consider

$$\begin{aligned} \boldsymbol{c}^\intercal \boldsymbol{x}^* - \boldsymbol{b}^\intercal \boldsymbol{y}^* &= \boldsymbol{c}^\intercal \boldsymbol{x}^* - (\boldsymbol{y}^*)^\intercal \boldsymbol{A}\boldsymbol{x}^* + (\boldsymbol{y}^*)^\intercal \boldsymbol{A}\boldsymbol{x}^* - \boldsymbol{b}^\intercal \boldsymbol{y}^* \\ &= (\boldsymbol{c}^\intercal - (\boldsymbol{y}^*)^\intercal \boldsymbol{A})\boldsymbol{x}^* + (\boldsymbol{y}^*)^\intercal(\boldsymbol{A}\boldsymbol{x}^* - \boldsymbol{b}) \\ &= \boldsymbol{r}^\intercal \boldsymbol{x}^* + \boldsymbol{s}^\intercal \boldsymbol{y}^* = \sum_{j=1}^{n} \boldsymbol{r}_j^* \boldsymbol{x}_j^* + \sum_{i=1}^{n} \boldsymbol{s}_i^* \boldsymbol{y}_i^* \end{aligned}$$

From the above it is obvious that (c') implies (c). On the other hand, (a) and (b) imply that $\boldsymbol{x}_j^*, \boldsymbol{r}_j^*$ $(j = 1, \dots, n)$ and $\boldsymbol{y}_i^*, \boldsymbol{s}_i^*$ $(i = 1, \dots, m)$ are all non-negative. Thus, (a)-(c) imply (c'). ■

It should be noted that the $\boldsymbol{y}_i^*$'s and the $\boldsymbol{r}_j^*$'s play a key role in sensitivity analysis of linear programming, where they usually referred to as *shadow prices* and *reduced cost* respectively.

These theoretical results are usually presented in the context of *duality theory* of linear programming. The main observation is that the boundedness certificate, as defined in (9)-(10), is a set of linear inequalities. Considering this set of inequalities as the feasibility set in terms of the variables $\boldsymbol{y}$, the *dual LP* of $\mathcal{P}$ is defined as[3]:

$$\begin{aligned} \mathcal{D}: \quad & \max \boldsymbol{b}^\intercal \boldsymbol{y} \\ \text{s.t.} \quad & \boldsymbol{A}^\intercal \boldsymbol{y} \le \boldsymbol{c} \\ & \boldsymbol{y} \ge \boldsymbol{0} \end{aligned}$$

Note that the constraints of the dual problem, $\mathcal{D}$, are identical to (9) and (10).

The relationship between the primal and the dual problems is symmetric as is evident from the following theorem.

Theorem 9 *The dual of the dual is the primal.*

Proof. First, we have to convert the dual ($\mathcal{D}$) to the format of the primal ($\mathcal{P}$). We'll do that by multiplying the objective function and the inequalities in (9) by -1, obtaining:

$$\begin{aligned} \mathcal{D}: \quad & \min -\boldsymbol{b}^\intercal \boldsymbol{y} \\ \text{s.t.} \quad & -\boldsymbol{A}^\intercal \boldsymbol{y} \ge -\boldsymbol{c} \\ & \boldsymbol{y} \ge \boldsymbol{0} \end{aligned}$$

[3]It is customary to call P, the *primal LP*

Taking the dual of the problem above, using $\boldsymbol{v}$ as its variables, we have:

$$\begin{aligned} \text{Dual of } \mathcal{D}: \quad & \max -\boldsymbol{c}^\mathsf{T}\boldsymbol{v} \\ \text{s.t.} \quad & -\boldsymbol{A}\boldsymbol{v} \leq -\boldsymbol{b} \\ & \boldsymbol{v} \geq \boldsymbol{0} \end{aligned}$$

Finally, we convert this problem by multiplying the objective function and the inequalities corresponding to $\boldsymbol{A}$ by -1. Thus we get the dual of $\mathcal{D}$ can be expressed as:

$$\begin{aligned} \text{Dual of } \mathcal{D}: \quad & \min \boldsymbol{c}^\mathsf{T}\boldsymbol{v} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{v} \geq \boldsymbol{b} \\ & \boldsymbol{v} \geq \boldsymbol{0} \end{aligned}$$

which is precisely the primal problem $\mathcal{P}$. ■

The major theorems of duality theory are basically reformulations of what we presented so far about the certificate, with replacing the statement "$\boldsymbol{y}$ satisfying (9) and (10)" with the statement "$\boldsymbol{y}$ is a feasible solution for $\mathcal{D}$". In particular, we have the following theorems.

Theorem 10 (Weak duality) *Suppose $\boldsymbol{x}$ and $\boldsymbol{y}$ are feasible solutions for the primal and the dual problems, respectively. Then, $\boldsymbol{c}^\mathsf{T}\boldsymbol{x} \geq \boldsymbol{b}^\mathsf{T}\boldsymbol{y}$.*

Note that the symmetry of the statement of the weak duality theorem with respect to the primal and dual problems implies that a dual (primal) feasible solution is a boundedness certificate to $\mathcal{P}$ ($\mathcal{D}$).

An obvious corollary for Theorem 10 is:

Corollary 11

(a) *If the dual is unbounded, then the primal is infeasible.*

(b) *If the primal is unbounded, then the dual is infeasible.*

(c) *Suppose $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ are feasible solutions for the primal and dual problems, respectively, and $\boldsymbol{c}^\mathsf{T}\boldsymbol{x}^* = \boldsymbol{b}^\mathsf{T}\boldsymbol{y}^*$. Then, $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ are optimal for the primal and dual problems, respectively.*

Next, we present the analog of Theorem 7.

Theorem 12 (Strong duality) *Suppose $\mathcal{P}$ and $\mathcal{D}$ are feasible. Then, there exist optimal solutions $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ for $\mathcal{P}$ and $\mathcal{D}$, respectively. In addition, for any such pair, $\boldsymbol{c}^\mathsf{T}\boldsymbol{x}^* = \boldsymbol{b}^\mathsf{T}\boldsymbol{y}^*$.*

Finally, we present the optimality conditions in terms of duality theory.

Optimality Conditions I

$\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ are optimal for $\mathcal{P}$ and $\mathcal{D}$, respectively, if and only if:

(a) $\boldsymbol{x}^*$ is feasible for the primal problem $\mathcal{P}$,

(b) $\boldsymbol{y}^*$ is feasible for the dual problem $\mathcal{D}$,

(c) $\boldsymbol{c}^\intercal \boldsymbol{x}^* = \boldsymbol{b}^\intercal \boldsymbol{y}^*$.

Optimality Conditions II

$\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ are optimal for $\mathcal{P}$ and $\mathcal{D}$, respectively, if and only if:

(a') $\boldsymbol{x}^*$ is feasible for the primal problem $\mathcal{P}$,

(b') $\boldsymbol{y}^*$ is feasible for the dual problem $\mathcal{D}$,

(c') $\boldsymbol{r}_j^* \boldsymbol{x}_j^* = 0 \ (j = 1, \ldots, n), \ \ \boldsymbol{s}_i^* \boldsymbol{y}_i^* = 0 \ (i = 1, \ldots, m)$.

(where $\boldsymbol{s}^* = \boldsymbol{A}\boldsymbol{x}^* - \boldsymbol{b}$ and $\boldsymbol{r}^* = \boldsymbol{c} - \boldsymbol{A}^\intercal \boldsymbol{y}^*$)

We summarize the duality theory results in the following table, displaying the status of $\mathcal{P}$ and $\mathcal{D}$ for any of the nine combinations of LP outcomes (existence of optimal solution, unboundedness, infeasibility) for both primal and dual.

	Primal feas.	**Primal unbd.**	**Primal infea.**
Dual feas.	Exist primal and dual optimal solutions with same o.f.v	Impossible	Impossible
Dual unbd.	Impossible	Impossible	possible
Dual infeas.	Impossible	Possible	Possible

Where the possibility of both the primal and dual problems to be infeasible is established by the following simple (one variable) example:

$$\mathcal{P} : \min -x \quad \text{s.t. } 0.x \geq 1, \ x \geq 0; \qquad \mathcal{D} : \max y \quad \text{s.t. } 0.y \leq -1, \ y \geq 0.$$