Feasibility and Infeasibility Certificates

Fall 2021

Plan

AMPL: matrix notation

Graphical Method

LP Forms

Certificates

Tennis betting problem

- x_1^i amount bet on i-th set for player I, $i = 1, 2, 3$.
- x_2^i amount bet on i-th set for player II, $i = 1, 2, 3$.
- y_1 amount bet on the match result for player I.
- y_2 amount bet on the match result for player II.
- w winnings.

$$
\begin{array}{lll}
\max & w & \\
\text{s.t.} & w & \leq \frac{2}{3}x_1^1 - x_2^1 + \frac{2}{3}x_1^2 - x_2^2 + \frac{2}{5}y_1 - y_2 \\
& w & \leq \frac{2}{3}x_1^1 - x_2^1 - x_1^2 + \frac{3}{2}x_2^2 + \frac{2}{3}x_1^3 - x_2^3 + \frac{2}{5}y_1 - y_2 \\
& w & \leq \frac{2}{3}x_1^1 - x_2^1 - x_1^2 + \frac{3}{2}x_2^2 - x_1^3 + \frac{3}{2}x_2^3 - y_1 + \frac{5}{2}y_2 \\
& w & \leq -x_1^1 + \frac{3}{2}x_2^1 - x_1^2 + \frac{3}{2}x_2^2 - y_1 + \frac{5}{2}y_2 \\
& w & \leq -x_1^1 + \frac{3}{2}x_2^1 + \frac{2}{3}x_1^2 - x_2^2 - x_1^3 + \frac{3}{2}x_2^3 - y_1 + \frac{5}{2}y_2 \\
& w & \leq -x_1^1 + \frac{3}{2}x_2^1 + \frac{2}{3}x_1^2 - x_2^2 + \frac{2}{3}x_1^3 - x_2^3 + \frac{2}{5}y_1 - y_2 \\
& \sum_{i=1}^{3}(x_1^i + x_2^i) + \sum_{i=1}^{2} y_i & \leq 100 \\
& x_1^i, x_2^i & \geq 0 \qquad i = 1, 2, 3 \\
& y_1, y_2 & \geq 0
\end{array}
$$

Code: ugly.mod

```
var x1a >=0;
var x2a >=0;
var x3a >=0;
var x1b >=0;
var x2b >=0;
var x3b >=0;
var ya >=0;
var yb >=0;

var w >=0;

maximize profit: w;

subject to budget: x1a+x2a+x3a+x1b+x2b+x3b+ya+yb <= 100;
subject to AA:  w <= 2/3*x1a - x1b + 2/3*x2a - x2b + 2/5*ya -yb ;
subject to ABA: w <= 2/3*x1a - x1b - x2a + 3/2*x2b + 2/3*x3a - x3b + 2/5*ya -yb ;
subject to ABB: w <= 2/3*x1a - x1b - x2a + 3/2*x2b - x3a + 3/2*x3b - ya + 5/2*yb;
subject to BB:  w <= -x1a + 3/2*x1b -x2a + 3/2*x2b - ya + 5/2*yb;
subject to BAB: w <= -x1a + 3/2*x1b + 2/3*x2a - x2b - x3a + 3/2*x3b - ya + 5/2*yb;
subject to BAA: w <= -x1a + 3/2*x1b + 2/3*x2a - x2b + 2/3*x3a - x3b + 2/5*ya -yb ;
```

Code: pretty.dat

```
set SCENS = AA ABA ABB BB BAB BAA ;
set VARS = x1a x2a x3a x1b x2b x3b ya yb;

param bud := 100;
param ones :=
  x1a 1
  x2a 1
  x3a 1
  x1b 1
  x2b 1
  x3b 1
  ya 1
  yb 1;

param matrix: x1a x2a x3a x1b x2b x3b ya yb :=
            AA 0.67 -1 0.67 -1 0 0 0.4 -1
            ABA 0.67 -1 -1 1.5 0.67 -1 0.4 -1
            ABB 0.67 -1 -1 1.5 -1 1.5 -1 2.5
            BB -1 1.5 -1 1.5 0 0 -1 2.5
            BAB -1 1.5 0.67 -1 -1 1.5 -1 2.5
            BAA -1 1.5 0.67 -1 0.67 -1 0.4 -1;
```

Code: pretty.mod

```
set SCENS;
set VARS;

param ones{VARS};
param matrix {SCENS, VARS};
param bud;

var x{VARS} >= 0;
var w >= 0;

maximize profit: w;

subject to budget: sum{i in VARS} ones[i]*x[i] <= bud;
subject to scen{j in SCENS}: w <= sum{i in VARS} matrix[j, i]*x[i];
```

Graphical method

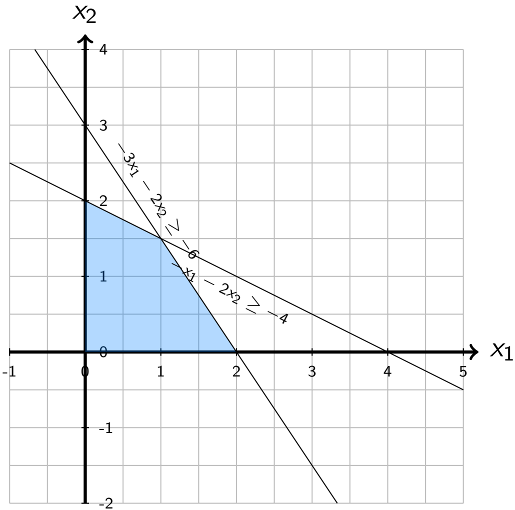
- Only for 2-variables toy examples
- Recipe:
  1. draw the convex polyhedron based on a set of constraints;
  2. draw the hyper-plane based on the objective

Example 1: Solve Graphically to Optimality

$$
\begin{array}{llllr}
\min & x_1 & +x_2 & & \\
s.t. & -x_1 & -2x_2 & \geq & -4 \\
 & -3x_1 & -2x_2 & \geq & -6 \\
 & x_1 & & \geq & 0 \\
 & & x_2 & \geq & 0
\end{array} \tag{1}
$$

Polyhedron

$$\begin{array}{llllll} \min & x_1 & +x_2 & & \\ s.t. & -x_1 & -2x_2 & \geq & -4 \\ & -3x_1 & -2x_2 & \geq & -6 \\ & x_1 & & \geq & 0 \\ & & x_2 & \geq & 0 \end{array}$$



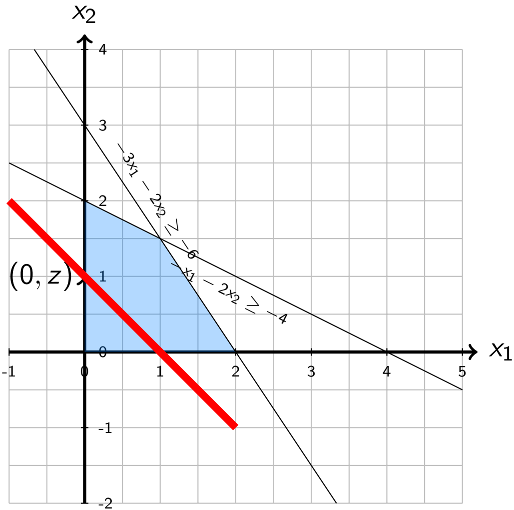
Objective

$$\begin{array}{lllll} \min & x_1 & +x_2 & & \\ s.t. & -x_1 & -2x_2 & \geq & -4 \\ & -3x_1 & -2x_2 & \geq & -6 \\ & x_1 & & \geq & 0 \\ & & x_2 & \geq & 0 \end{array}$$

Consider the objective by $z = x_1 + x_2$ which can be rewritten as

$$x_2 = -x_1 + z.$$

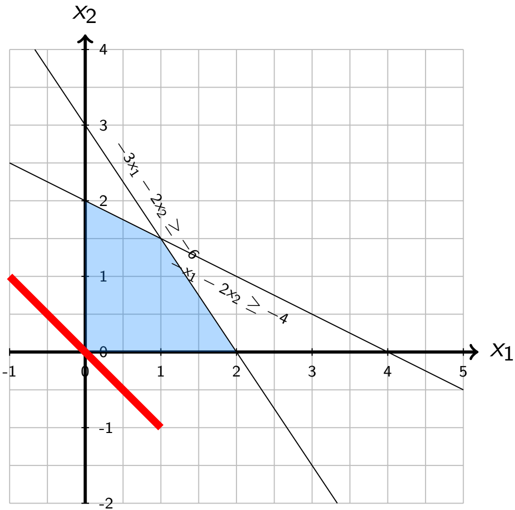
This is a line with slope -1 and intercept z.



Solution

$$\begin{array}{llllll} \min & x_1 & +x_2 & & \\ s.t. & -x_1 & -2x_2 & \geq & -4 \\ & -3x_1 & -2x_2 & \geq & -6 \\ & x_1 & & \geq & 0 \\ & & x_2 & \geq & 0 \end{array}$$

We move the line $x_2 = -x_1 + z$ for minimizing its intercept z. From the graph, the solution is $(0, 0)$.



LP Forms

- **Symmetric form:**

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax \geq b, \;\; x \geq 0. \end{aligned}$$

- **Standard form:**

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax = b, \;\; x \geq 0. \end{aligned}$$

- **Inequality form:**

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax \geq b. \end{aligned}$$

Example:

$$\begin{aligned} \min \quad & |x_1| + |x_2| \\ \text{s.t.} \quad & |x_1 - x_2| \leq 2 \end{aligned}$$

To do: 1) symmetric form, 2) standard form, 3) inequality form.

Example

Reformulation to Symmetric form:

$$
\begin{aligned}
\min \quad & x_1^+ + x_1^- + x_2^+ + x_2^- \\
-x_1^+ + x_1^- + x_2^+ - x_2^- &\geq -2 \\
x_1^+ - x_1^- - x_2^+ + x_2^- &\geq -2 \\
x_1^+ \geq 0, \quad x_1^- \geq 0, \quad x_2^+ \geq 0, \quad x_2^- &\geq 0
\end{aligned}
$$

Example

Reformulation to Standard form:

$$
\begin{aligned}
\min \quad & x_1^+ + x_1^- + x_2^+ + x_2^- \\
& x_1^+ - x_1^- - x_2^+ + x_2^- + s_1 = 2 \\
& -x_1^+ + x_1^- + x_2^+ - x_2^- + s_2 = 2 \\
& x_1^+ \geq 0, \quad x_1^- \geq 0, \quad x_2^+ \geq 0, \quad x_2^- \geq 0, \quad s_1 \geq 0, \quad s_2 \geq 0.
\end{aligned}
$$

Example

Reformulation to Inequality form:

$$
\begin{aligned}
\min \quad & x_1^+ + x_1^- + x_2^+ + x_2^- \\
-x_1^+ + x_1^- + x_2^+ - x_2^- &\geq -2 \\
+x_1^+ - x_1^- - x_2^+ + x_2^- &\geq -2 \\
x_1^+ &\geq 0 \\
x_1^- &\geq 0 \\
x_2^+ &\geq 0 \\
x_2^- &\geq 0.
\end{aligned}
$$

A fact to support the Lecture

For all $A \in \mathbb{R}^{n \times m}$ and $\mathbf{b}, \mathbf{y} \in \mathbb{R}^n$

$$A\mathbf{x} = \mathbf{b} \qquad \implies \qquad \mathbf{y}^T A\mathbf{x} = \mathbf{y}^T \mathbf{b}$$

A fact to support the Lecture

For all $A \in \mathbb{R}^{n \times m}$ and $\mathbf{b}, \mathbf{y} \in \mathbb{R}^n$

$$A\mathbf{x} = \mathbf{b} \quad \implies \quad \mathbf{y}^T A\mathbf{x} = \mathbf{y}^T \mathbf{b}$$

$$A\mathbf{x} \geq \mathbf{b} \quad \implies \quad \begin{cases} \mathbf{y}^T A\mathbf{x} \geq \mathbf{y}^T \mathbf{b} \\ \mathbf{y} \geq 0 \end{cases}$$

A fact to support the Lecture

For all $A \in \mathbb{R}^{n\times m}$ and $\mathbf{b}, \mathbf{y} \in \mathbb{R}^n$

$$A\mathbf{x} = \mathbf{b} \quad \Longrightarrow \quad \mathbf{y}^T A\mathbf{x} = \mathbf{y}^T\mathbf{b}$$

$$A\mathbf{x} \geq \mathbf{b} \quad \Longrightarrow \quad \begin{cases} \mathbf{y}^T A\mathbf{x} \geq \mathbf{y}^T\mathbf{b} \\ \mathbf{y} \geq 0 \end{cases}$$

What if we drop $\mathbf{y} \geq 0$? — Counterexample

$$\begin{bmatrix} 3 & 1 \\ -1 & 6 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} \geq \begin{bmatrix} 10 \\ -4 \end{bmatrix} \quad \not\Rightarrow \quad [-1 \;\; 1] \begin{bmatrix} 3 & 1 \\ -1 & 6 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} \geq [-1 \;\; 1] \begin{bmatrix} 10 \\ -4 \end{bmatrix}$$

$$\text{since} \begin{cases} 12 \geq 10 \\ -4 \geq -4 \end{cases} \quad \not\Rightarrow \quad -16 \geq -14$$

Feasibility and Infeasibility Certificates

- Finding certificates is not an easy problem in general.
- If I found a certificate, which is a vector that satisfies the corresponding condition, then something holds. Not necessarily the other way around. LP is great because the other way around also holds.

Certificate of Feasibility/Infeasibility

- *Symmetric form:* Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$,

$$\min \ \mathbf{c}^\top \mathbf{x} \quad \text{s.t. } A\mathbf{x} \geq \mathbf{b}, \ \mathbf{x} \geq 0.$$

- *Certificate of feasibility* is $\mathbf{z} \in \mathbb{R}^n$ that satisfies

$$A\mathbf{z} \geq \mathbf{b}, \quad \mathbf{z} \geq 0.$$

- *Certificate of infeasibility* is $\mathbf{w} \in \mathbb{R}^m$ that satisfies

$$(\mathbf{w}^\top A)^\top = A^\top \mathbf{w} \leq 0, \quad \mathbf{w} \geq 0, \quad \mathbf{w}^\top b > 0.$$

Example 1:

$$\begin{aligned} \min \quad 5x_1 - 6x_2 + 4x_3 \\ x_1 - x_2 - 4x_3 \geq 1 \\ -x_1 + 3x_2 - x_3 \geq 2 \\ x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0 \end{aligned}$$

Example 1:

$$\begin{aligned} \min \quad & 5x_1 - 6x_2 + 4x_3 \\ & x_1 - x_2 - 4x_3 \geq 1 \\ & -x_1 + 3x_2 - x_3 \geq 2 \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0 \end{aligned}$$

Certificate of Feasibility

$$\begin{aligned} a_{11}z_1 + a_{12}z_2 + a_{13}z_3 &\geq b_1, \\ a_{21}z_1 + a_{22}z_2 + a_{23}z_3 &\geq b_2, \\ z_1, z_2, z_3 &\geq 0. \end{aligned}$$

We can find the solution, i.e., $(z_1, z_2, z_3) = (3, 2, 0)$. So the LP is feasible.

Example 1:

Certificate of Infeasibility

$$\begin{aligned} \min \quad 5x_1 - 6x_2 + 4x_3 \\ x_1 - x_2 - 4x_3 \geq 1 \\ -x_1 + 3x_2 - x_3 \geq 2 \\ x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0 \end{aligned}$$

$$\begin{aligned} b_1 w_1 + b_2 w_2 &> 0, \\ a_{11} w_1 + a_{21} w_2 &\leq 0, \\ a_{12} w_1 + a_{22} w_2 &\leq 0, \\ a_{13} w_1 + a_{23} w_2 &\leq 0, \\ w_1, w_2 &\geq 0. \end{aligned}$$

Example 1:

Certificate of Infeasibility

$$
\begin{aligned}
w_1 + 2w_2 &> 0, \\
w_1 - w_2 &\leq 0, \\
-w_1 + 3w_2 &\leq 0, \\
-4w_1 - w_2 &\leq 0, \\
w_1, w_2 &\geq 0.
\end{aligned}
$$

However, no solution exists. Why?

$$w_1 - w_2 \leq 0, \ -w_1 + 3w_2 \leq 0, \ w_1 \geq 0$$
$$\Longrightarrow \quad w_1 = 0,$$

and

$$w_1 - w_2 \leq 0, \ -w_1 + 3w_2 \leq 0, \ w_2 \geq 0$$
$$\Longrightarrow \quad w_2 = 0.$$

but $w_1 + 2w_2 > 0 \ \Rightarrow \ (w_1, w_2) \neq (0, 0)$.

Example 2:

$$
\begin{array}{lllll}
\min & 2x_1 & +x_2 & & \\
\text{s.t.} & x_1 & +2x_2 & \geq & 4 \\
 & 3x_1 & +2x_2 & \geq & 6 \\
 & x_1 & & \geq & 0 \\
 & & x_2 & \geq & 0
\end{array}
$$

Certificate of Feasibility

$$
\begin{aligned}
a_{11}z_1 + a_{12}z_2 &\geq b_1, \\
a_{21}z_1 + a_{22}z_2 &\geq b_2, \\
z_1, z_2 &\geq 0.
\end{aligned}
$$

We can find the solution, i.e., $(z_1, z_2) = (2, 1)$. So the LP is feasible.

Example 3:

$$
\begin{array}{llllll}
\min & x_1 & +x_2 & & \\
\text{s.t.} & x_1 & & \geq & 6 \\
 & & x_2 & \geq & 6 \\
 & -x_1 & -x_2 & \geq & -10
\end{array}
$$

Example 3:

$$
\begin{array}{llllll}
\min & x_1 & +x_2 & & \\
\text{s.t.} & x_1 & & \geq & 6 \\
& & x_2 & \geq & 6 \\
& -x_1 & -x_2 & \geq & -10
\end{array}
$$

or in Symmetric form

$$
\begin{array}{llllll}
\min & x_1 & +x_2 & & \\
\text{s.t.} & x_1 & & \geq & 6 \\
& & x_2 & \geq & 6 \\
& -x_1 & -x_2 & \geq & -10 \\
& x_1 & & \geq & 0 \\
& & x_2 & \geq & 0
\end{array}
$$

Example 3:

$$\begin{array}{llllll} \min & x_1 & +x_2 & & \\ \text{s.t.} & x_1 & & \geq & 6 \\ & & x_2 & \geq & 6 \\ & -x_1 & -x_2 & \geq & -10 \end{array}$$

or in Symmetric form

$$\begin{array}{llllll} \min & x_1 & +x_2 & & \\ \text{s.t.} & x_1 & & \geq & 6 \\ & & x_2 & \geq & 6 \\ & -x_1 & -x_2 & \geq & -10 \\ & x_1 & & \geq & 0 \\ & & x_2 & \geq & 0 \end{array}$$

Certificate of Infeasibility

$$\begin{array}{rcl} b_1 w_1 + b_2 w_2 + b_3 w_3 & > & 0, \\ a_{11} w_1 + a_{21} w_2 + a_{31} w_3 & \leq & 0, \\ a_{12} w_1 + a_{22} w_2 + a_{32} w_3 & \leq & 0, \\ w_1, w_2, w_3 & \geq & 0. \end{array}$$

and

$$\begin{array}{lllll} 6w_1 & +6w_2 & -10w_3 & > & 0, \\ w_1 & & -w_3 & \leq & 0, \\ & +w_2 & -w_3 & \leq & 0, \\ w_1, & w_2, & w_3 & \geq & 0. \end{array}$$

A solution is $(w_1, w_2, w_3) = (1, 1, 1)$.

Thank you for your attention !