EE650 Linear System Theory
Problem Set 1

Issued 21 Aug 2023; Due 22 Sep 2023, 14:00 HST

Problem 1

Functions. Consider $f : \mathbb{R}^3 \to \mathbb{R}^3$, defined as

$$f(x) = Ax, A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, x \in \mathbb{R}^3$$

Is f a function? Is it injective? Is it surjective? Justify your answers.

Problem 2

Functions. You work at a chemical plant where you need to keep track of the temperatures of three chemical reactors. Your coworker designed a user interface that takes in $x \in \mathbb{R}^3$, where x_j is the temperature of reactor j. The user interface then maps x through $f(x)$, and outputs the values $y \in \mathbb{R}^2$.

$$y = f(x) = Ax$$
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

1. Is f a function? What is the domain? What is the codomain? Is it injective? Is it surjective?
2. Is this a well-designed interface? In other words, does it matter if this is a function/injective/surjective? Why or why not?

Problem 3

Fields.

1. Define addition and multiplication on $\{0, 1\}$ to form a field. Show that your result is a field.
2. Is GL_n, the set of all $n \times n$ nonsingular matrices, a field? Justify your answer.

Problem 4

Vector Spaces.

1. Show that $(\mathbb{R}^n, \mathbb{R})$, the set of all ordered n-tuples of elements from the field of real numbers $\mathbb{R}$, is a vector space.
2. Show that the set of all polynomials in s of degree k or less with real coefficients is a vector space over the field $\mathbb{R}$. Find a basis. What is the dimension of the vector space?

Problem 5

Subspaces. Is a plane in $\mathbb{R}^3$ a subspace of $\left(\mathbb{R}^3, \mathbb{R}\right)$?

Problem 6

Subspaces. Suppose $U_1, U_2, \ldots, U_m$ are subspaces of a vector space V. The sum of $U_1, U_2, \ldots, U_m$, denoted $U_1 + U_2 + \ldots + U_m$, is defined to be the set of all possible sums of elements of $U_1, U_2, \ldots, U_m$:

$$U_1 + U_2 + \ldots + U_m = \{u_1 + u_2 + \ldots + u_m : u_1 \in U_1, \ldots, u_m \in U_m\}$$

1. Is $U_1 + U_2 + \ldots + U_m$ a subspace of V ?
2. Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that $U_1 + W = U_2 + W$, then $U_1 = U_2$. Problem 7: Subspaces. Consider the set of sequences $\{f_k\}_{k=0}^{\infty} := \{f_0, f_1, f_2, \ldots\}$ satisfying $f_k = f_{k-1} + f_{k-2}$ where f_0 and f_1 are arbitrary real numbers. Is this a subspace in the vector space of all sequences of real numbers over the field of real numbers?

Problem 8

Subspaces. Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Problem 9

Linear Independence. Let V be the set of 2-tuples whose entries are complex-valued rational functions. Consider two vectors in V :

$$v_1 = \begin{bmatrix} 1/(s+1) \\ 1/(s+2) \end{bmatrix}, v_2 = \begin{bmatrix} (s+2)/((s+1)(s+3)) \\ 1/(s+3) \end{bmatrix}$$

Is the set $\{v_1, v_2\}$ linearly independent over the field of rational functions? Is it linearly independent over the field of real numbers?

Problem 10

Linear Independence. Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

Is the set $\left\{I, A, A^2\right\}$ linearly dependent or independent in $\mathbb{R}^{2\times 2}$?

Problem 11

Linear Independence. Which of the following sets are linearly independent in $\mathbb{R}^3$?

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}$$

Problem 12

Bases. Let U be the subspace of $\mathbb{R}^5$ defined by

$$U = \left\{ [x_1, x_2, \ldots, x_5]^T \in \mathbb{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4 \right\}$$

Find a basis for U.

Problem 13

Bases. Prove that if $\{v_1, v_2, \ldots v_n\}$ is linearly independent in V, then so is the set $\{v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n\}$.

Problem 14

Linearity. Are the following maps $\mathcal{A}$ linear?

1. $\mathcal{A}(u(t)) = u(-t)$ for $u(t)$ a scalar function of time
2. How about $y(t) = \mathcal{A}(u(t)) = \int_0^t e^{-\sigma} u(t - \sigma) d\sigma$?
3. How about the map $\mathcal{A} : as^2 + bs + c \to \int_0^s (bt + a)dt$ from the space of polynomials with real coefficients to itself?

Problem 15

Range, Nullspace of linear maps. Consider a linear map $\mathcal{A}$. Prove that $R(\mathcal{A})$ and $N(\mathcal{A})$ are subspaces.

Problem 16

Rank-Nullity Theorem. Let $\mathcal{A}$ be a linear map from U to V with dim $U = n$ and dim$V = m$. Show that

$$\dim\, R(\mathcal{A}) + \dim\, N(A) = n$$

Problem 17

Properties of Linear maps. Consider the linear map $\mathcal{A} : U \to V$. Suppose that $\mathcal{A}(x_0) = b$. Then $\mathcal{A}(x) = b$ iff $x - x_0 \in N(\mathcal{A})$.

Problem 18

Matrix Representation of a Linear Map. Let $\mathcal{A} : (F^2, F) \to (F^3, F)$ be defined by $\mathcal{A}(x, y) = (x + 3y, 2x + 5y, 7x + 9y)$. What is the matrix representation of $\mathcal{A}$ with respect to the standard bases, and with respect to new bases:

$$B_{F^2} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}; B_{F^3} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Problem 19

Matrix Representation of a Linear Map. Let $\mathcal{A} : (U, F) \to (V, F)$ with $\dim U = n$ and $\dim V = m$ be a linear map with rank $(\mathcal{A}) = k$. Show that there exist bases $\{u_i\}_{i=1}^n$ and $\{v_j\}_{j=1}^m$ of U and V respectively such that with respect to these bases $\mathcal{A}$ is represented by the block diagonal matrix

$$A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

What are the dimensions of the different blocks?

Problem 20

Matrix Representation of a Linear Map. From Callier and Desoer, p.421

Let A be a linear map of the n-dimensional linear space (V, F) onto itself. Assume that for some $\lambda \in F$ and basis $\{v_i\}_{i=1}^n$ we have

$$Av_1 = \lambda v_1$$

and

$$Av_k = \lambda v_k + v_{k-1}, k = 2, \ldots, n$$

Obtain a representation of A with respect to this basis.

Problem 21

Sylvester's Inequality. We introduced the rank of a linear map as the dimension of its range. Since all linear maps between finite dimensional vector spaces can be represented with matrices, it is important to note that the rank of a linear map also equals the matrix-rank of its matrix representation. (Can be proven as an exercise)

Given $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times p}$ show that

$$\text{rank }(A) + \text{rank }(B) - n \leq \text{rank }(AB) \leq \min\{\text{rank }(A), \text{rank }(B)\}$$

Problem 22

Prove that the induced matrix norm $\ell_2 \to \ell_2$ of a matrix is the largest singular value of the matrix:

$$\text{if } A : (U, \ell_2) \to (V, \ell_2) \text{ then } \|A\|_i = \bar{\sigma}(A) = \sigma_{\max}(A)$$

Problem 23

Consider an inner product space V, with $x, y \in V$. Show, using properties of the inner product, that

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

where $\|\cdot\|$ is the norm induced by the inner product.

Problem 24

Adjoints. Consider an inner product space $(\mathbb{C}^n, \mathbb{C})$, equipped with the standard inner product in $\mathbb{C}^n$, and a map $\mathcal{A} : \mathbb{C}^n \to \mathbb{C}^n$, which consists of matrix multiplication by an $n \times n$ matrix A. Find the adjoint of $\mathcal{A}$.

Problem 25

Adjoints. Suppose that $\mathcal{A} : V \to W$ is a linear map, and V and W are two inner product spaces. Prove that the adjoint map $\mathcal{A}^* : W \to V$ is linear

Problem 26

Adjoints. Consider a linear map $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^n$, defined by:

$$\mathcal{A} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

What is $\mathcal{A}^*$?

Problem 27

Singular values. You are given a matrix $A \in \mathbb{R}^{n \times n}$, and you are told it has singular values $\sigma_1 > \sigma_2 > \ldots > \sigma_n > 0$, but you are not told what these singular values are. You wish to develop a simple interactive calculation to estimate the largest singular value σ_1.

Consider the following interactive calculation:

- Choose $x_0 \in \mathbb{R}^n$
- Calculate $x_{i+1} = A^* A x_i, i = 0, 1, 2, \ldots$
- Consider the sequence:

$$\frac{\|x_1\|_2^2}{\|x_0\|_2^2}, \frac{\|x_2\|_2^2}{\|x_1\|_2^2}, \frac{\|x_3\|_2^2}{\|x_2\|_2^2}, \ldots$$

The claim is that the sequence converges to σ_1. Determine whether or not the claim is true depending on the choice of x_0.