EE650 Linear System Theory
Problem Set 2

Issued 18 Sep 2023; Due 20 Oct 2023, 14:00 HST

Problem 1

Local or global Lipschitz condition. Consider the pendulum equation with friction and constant input torque:

$$\dot{x_1} = x_2 \dot{x_2} = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 + \frac{T}{ml^2},$$

where x_1 is the angle that the pendulum makes with the vertical, x_2 is the angular rate of change, m is the mass of the bob, l is the length of the pendulum, k is the friction coefficient, and T is a constant torque. Let $B_r = \{x \in \mathbb{R}^2 : \|x\| < r\}$. For this system (represented as $\dot{x} = f(x)$) find whether f is locally Lipschitz in x on B_r for sufficiently small r, locally Lipschitz in x on B_r for any finite r, or globally Lipschitz in x (i.e. Lipschitz for all $x \in \mathbb{R}^2$).

Problem 2

Existence and uniqueness of solutions to differential equations. Consider the following two systems of differential equations:

$$\dot{x_1} = -x_1 + e^t \cos(x_1 - x_2)\dot{x_2} = -x_2 + 15\sin(x_1 - x_2)$$

and

$$\dot{x_1} = -x_1 + x_1 x_2 \dot{x_2} = -x_2$$

1. Do they satisfy a global Lipschitz condition?
2. For the second system, your friend asserts that the solutions are uniquely defined for all possible initial conditions, and they all tend to zero for all initial conditions. Do you agree or disagree?

Problem 3

Perturbed nonlinear systems. Suppose that some physical system obeys the differential equation

$$\dot{x} = p(x,t), x(t_0) = x_0, \forall t \geq t_0,$$

where $p(\cdot,\cdot)$ obeys the conditions of the fundamental theorem. Suppose that as a result of some perturbation the equation becomes

$$\dot{z} = p(x,t) + f(t), z(t_0) = x_0 + \delta x_0, \forall t \geq t_0$$

Given that for $t \in [t_0, t_0 + T]$, $\|f(t)\| \leq \varepsilon_1$ and $\|\delta x_0\| \leq \varepsilon_0$, find a bound on $\|x(t) - z(t)\|$ valid for $t \in [t_0, t_0 + T]$.

Problem 4

Dynamical systems, time invariance. Suppose that the output of a system is represented by

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} u(\tau) d\tau$$

Is the system time invariant? You may select the input space $\mathcal{U}$ to be the set of bounded, piecewise continuous, real-valued functions defined on $(-\infty, \infty)$.

Problem 5

Solution of a matrix differential equation. Let $A_1(\cdot)$, $A_2(\cdot)$ and $F(\cdot)$, be known piecewise continuous $n \times n$ matrix-valued functions. Let Φ_i be the transition matrix of $\dot{x} = A_i(t)x$ for $i = 1, 2$. Show that the solution of the matrix differential equation:

$$\dot{X}(t) = A_1(t)X(t) + X(t)A_2^T(t) + F(t), X(t_0) = X_0$$

is

$$X(t) = \Phi_1(t, t_0)X_0\Phi_2^T(t, t_0) + \int_{t_0}^{t} \Phi_1(t, \tau)F(\tau)\Phi_2^T(t, \tau)d\tau$$

Problem 6

Satellite Problem, linearization, state space model. Model the earth and a satellite as particles. The *normalized* equations of motion, in an earth-fixed inertial frame, simplified to 2 dimensions (from Lagrange's equations of motion, the Lagrangian $L = T - V = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 - \frac{k}{r}$):

$$\ddot{r} = r\dot{\theta}^2 - \frac{k}{r^2} + u_1$$

$$\ddot{\theta} = -2\frac{\dot{\theta}}{r}\dot{r} + \frac{1}{r}u_2$$

with u_1, u_2 representing the radial and tangential forces due to thrusters. The reference orbit with $u_1 = u_2 = 0$ is circular with $r(t) \equiv p$ and $\theta(t) = \omega t$. From the first equation, it follows that $p^3\omega^2 = k$. Obtain the linearized equation describing this orbit.

Problem 7

State Transition Matrix, calculations. Calculate the state transition matrix for $\dot{x}(t) = A(t)x(t)$, with the following $A(t)$:

1. $A(t) = \begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix}$
2. $A(t) = \begin{bmatrix} -2t & 0 \\ 1 & -1 \end{bmatrix}$
3. $A(t) = \begin{bmatrix} 0 & \omega(t) \\ -\omega(t) & 0 \end{bmatrix}$
4. For systems 1. and 2. above, describe the zero input (non-zero initial state) response.

Hint: for part 3. above, let $\Omega(t) = \int_0^t \omega(t')dt'$; and consider the matrix

$$\begin{bmatrix} \cos\Omega(t) & \sin\Omega(t) \\ -\sin\Omega(t) & \cos\Omega(t) \end{bmatrix}$$

Problem 8

Sampled Data System. You are given a linear, time-invariant system

$$\dot{x} = Ax + Bu$$

which is sampled every T seconds. Denote $x(kT)$ by $x(k)$. Further, the input u is held constant between kT and $(k+1)T$, that is, $u(t) = u(k)$ for $t \in [kT, (k+1)T]$. Derive the exact state equation for the sampled data system, that is, give a formula for $x(k+1)$ in terms of $x(k)$ and $u(k)$.

Problem 9

Discrete-time LQR. Consider the following optimal control problem where we are interested in controlling the output instead of state:

$$\min_U \sum_{\tau=0}^{N-1} (y_\tau^T Q y_\tau + u_\tau^T R u_\tau)$$

subject to

$$\begin{aligned} x_{t+1} = Ax_t + Bu_t, t \in \{0, 1, \dots, N-1\} \\ y_t = Cx_t \\ x_0 = x^{init} \end{aligned}$$

Here, U is the sequence of control inputs. Find an LQR-like sequence of matrix updates that computes the optimal cost-to-go at all times and the optimal feedback controller at all times.

Problem 10

Discrete-time LQR. In the above problem, suppose the system matrices are given by:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

and the cost matrices are given by $Q = Q_f = \rho_Q I$ and $R = \rho_R I$. Let $x_0 = (1, 0)$ and $N = 20$. Explain how the output, control and cost-to-go change under the optimal feedback when

1. $\rho_Q = 1, \rho_R = 1$,
2. $\rho_Q = 10^3, \rho_R = 1$,
3. $\rho_Q = 1, \rho_R = 10^3$.

You can use MATLAB, python, or any other tools you like to solve the problem.

Problem 11

Discrete-time LQR. Suppose that we would like the system to track a reference state trajectory. Derive the optimal LQR control policy and the cost-to-go function for the reference trajectory problem for LTI systems when the reference trajectory (x^*, u^*) is *not* dynamically feasible, meaning that: $x^*_{t+1} \neq f(x^*_t, u^*_t)$.

Problem 12

Continuous-time LQR, infinite horizon. Consider the system described by the equations $\dot{x} = Ax + Bu, y = Cx$, where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

1. Determine the optimal control $u^*(t) = F^*x(t), t > 0$ which minimizes the performance index

   $$J = \int_0^\infty (y^2(t) + \rho u^2(t))dt$$

   where ρ is positive and real.

2. Observe how the eigenvalues of the dynamic matrix of the resulting closed loop system change as a function of ρ. Can you comment on the results?