EE650 Linear System Theory
Problem Set 4

Issued 6 Nov 2023; Due 8 Dec 2023 14:00 HST

Problem 1

Adjoint of Observability Map. Given the observability map $\mathcal{L}_O : \mathbb{R}^n \rightarrow \mathcal{Y}_{[t_0,t_1]}$ where $\mathcal{L}_O(x_0) = C(\cdot)\Phi(\cdot, t_0)x_0$, derive its adjoint map $\mathcal{L}_O^*$.

Problem 2

Controllability over time intervals. Given a linear time varying system $R(\cdot) = [A(\cdot), B(\cdot), C(\cdot), D(\cdot)]$, show that if $R(\cdot)$ is completely controllable on $[t_0, t_1]$ then $R(\cdot)$ is completely controllable on any $[t_0', t_1']$, where $t_0' \leq t_0 < t_1 \leq t_1'$. Show that this is no longer true when the interval $[t_0, t_1]$ is not a subset of $[t_0', t_1']$.

Problem 3

Controllability, characteristic and minimal polynomials. Consider an LTI system (A, B, C) with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_i}$, $C \in \mathbb{R}^{n_o\times n}$. You are told that the characteristic polynomial of A is $(s - \lambda_1)^{d_1}(s - \lambda_2)^{d_2} \cdots (s - \lambda_k)^{d_k}$ and the minimal polynomial is $(s - \lambda_1)^{m_1}(s - \lambda_2)^{m_2} \cdots (s - \lambda_k)^{m_k}$. The ordering of the λ_i is chosen to be such that $m_1 \leq m_2 \leq \ldots \leq m_k$. Compute the minimum number of inputs required (that is the minimum size of n_i) so as to make the pair (A, B) completely controllable. Similarly, compute the minimum value of n_o to make the pair (A, C) completely observable.

Problem 4

Observability Tests for LTI Systems. Consider the following theorem:

Theorem 1 *Consider an LTI system (A, B, C) with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_i}$, $C \in \mathbb{R}^{n_o\times n}$. The following statements are equivalent:*

1. *The LTI system represented by (A, C) is completely observable on some $[0, \Delta]$*
2. *rank* $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$
3. *rank* $\begin{bmatrix} sI - A \\ C \end{bmatrix} = n, \forall s \in \sigma(A)$

Prove the following 4 directions:

- 1. $\Rightarrow$ 2.
- 2. $\Rightarrow$ 1.

- 2. $\Rightarrow$ 3.
- 3. $\Rightarrow$ 2.

Hint: One way to prove these is to consider the matrices (A^T, C^T) *and follow the controllability results directly*

Problem 5

A prequel question to controller design. Consider the linear time invariant system with state equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_3 & -\alpha_2 & -\alpha_1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

Insert state feedback: the input to the overall closed loop system is v and $u = v - k^T x$ where k is a constant row vector. Show that given *any* polynomial $p(s) = \sum_{k=0}^{3} b_k s^{3-k}$ with $b_0 = 1$, there is a row vector k such that the closed loop system has $p(s)$ as its characteristic equation. (This naturally extends to n dimensions and implies that any system with a representation that can be put into the form above, called Controllable Canonical Form, can be stabilized by state feedback.)

Problem 6

State vs. Output Feedback. Consider a dynamical system described by:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 7 & -4 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 3 \end{bmatrix}$$

For each of cases below, derive a state space representation of the resulting closed loop system, and determine the characteristic equation of the resulting closed loop "A" matrix (called the closed loop characteristic equation):

1. $u = -[f_1 \; f_2]x$
2. $u = -ky$ (here $k \in \mathbb{R}$)

Problem 7

Feedback control design by eigenvalue placement. Consider the dynamic system:

$$\frac{d^4\theta}{dt^4} + \alpha_1 \frac{d^3\theta}{dt^3} + \alpha_2 \frac{d^2\theta}{dt^2} + \alpha_3 \frac{d\theta}{dt} + \alpha_4 \theta = u$$

where u represents an input force, α_i are real scalars. Assuming that $\frac{d^3\theta}{dt^3}$, $\frac{d^2\theta}{dt^2}$, $\frac{d\theta}{dt}$ and θ can all be measured, design a state feedback control scheme which places the closed-loop eigenvalues at $s_1 = -1$, $s_2 = -1$, $s_3 = -1 + j1$, $s_4 = -1 - j1$.

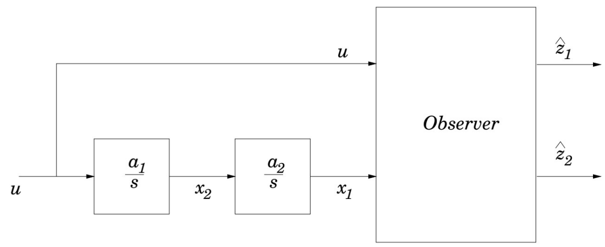

Figure 1: Simple model of a DC Servo system

Problem 8

Observer design. Figure shows a block diagram representation of a simple model of a DC servo system: x_1 is a voltage signal proportional to the output angular velocity x_2.

1. Design a full order observer, with observer gain matrix T given by
$$T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix},$$
for x_1 and x_2 so that the characteristic polynomial associated with the error dynamics is given by:
$$\Delta_e(s) = s^2 + 2\zeta_e\omega_e s + \omega_e^2$$
("Design" means to write down the equations for the observer, with expressions for gains T_1 and T_2.)

2. Now, the observer is a system with inputs u and x_1, and outputs $\hat{z}_1$ and $\hat{z}_2$. Thus, there are four possible transfer functions between inputs and outputs - these may be included as elements in a 2×2 matrix. Evaluate the followint *matrix of transfer functions* $M(s)$ between the inputs to the observer u and x_1, and its outputs $\hat{z}_1$ and $\hat{z}_2$:
$$M(s) = \begin{bmatrix} \hat{z}_1(s)/u(s) & \hat{z}_1(s)/x_1(s) \\ \hat{z}_2(s)/u(s) & \hat{z}_2(s)/x_1(s) \end{bmatrix}$$
as a function of gains T_1 and T_2, as well as system parameters a_1 and a_2.

3. Now determine $M(s)$ as $T_2 \to \infty$. Discuss the meaning of the result.

Problem 9

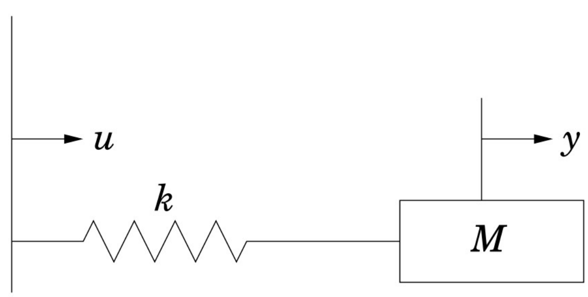

Figure 2: Simple robotic arm

Control of a Flexible Robot Arm. A simplified model for the control of a flexible robotic arm is shown in Figure . Here, k is a spring constant which models the flexibility of the arm, M represents the mass of the arm, y, the output, is the mass position, and u, the input, is the position of the end of the spring. Here, $k/M = 900$ rad/s^2.

The equations of motion for this system are thus given by $M\ddot{y} + k(y - u) = 0$. Define state variables $x_1 = y$, $x_2 = \dot{y}$.

1. Write the equations of motion in state space form. Where are the open loop eigenvalues?

2. Design a full state observer with observer eigenvalues at $s = -100 \pm 100j$.

3. Could both state-variables of the system be estimated if only a measurement of $\dot{y}$ were available?

4. Design a state feedback controller with gain matrix F giving the closed loop system roots at $s = -20 \pm 20j$.

5. Would it be reasonable to design a control law for the system with roots at $s = -200 \pm 200j$? Explain why or why not.