EE650 Linear System Theory
Final

Your answers must be supported by analysis, proof, or counterexample. You may quote the results that we derived in class in your solutions. There are 8 questions: Please make sure your exam paper has all 8 questions. Approximate points for each question are indicated.

You are allowed to use one 8.5 × 11 cheat sheet (both sides).

Problem 1

Linear Quadratic Regulator (8 points). Consider the following discrete linear time-invariant dynamical system:

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t, t \in \{0, 1, \ldots, N-1\} \\ x_0 &= x^{init}, \end{aligned}$$

and the cost function:

$$J(U, x_0) = \sum_{\tau=0}^{N-1} \big((x_\tau - x^*)^T Q (x_\tau - x^*) + (u_\tau - u^*)^T R (u_\tau - u^*)\big) + (x_N - x^*)^T Q_f (x_N - x^*)$$

where $Q, R, Q_f \succ 0$ are positive definite matrices and $U := (u_0, u_1, \ldots, u_{N-1})$. We say a reference point (x^*, u^*) is dynamically feasible if it satisfies the system dynamics that is:

$$x^* = Ax^* + Bu^*,$$

otherwise, the reference point is called dynamically infeasible.

Suppose that the reference point (x^*, u^*) is dynamically feasible. Derive the optimal LQR control policy and cost-to-go function that minimize $J(U, x_0)$ subject to the system dynamics.

Problem 2

Cayley-Hamilton Theorem (4 points). Consider the following matrix:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

1. Find the characteristic polynomial of A.
2. Express A^4 in terms of the lowest degree polynomial in A.
3. Using Cayley-Hamilton theorem, show that $e^{At} = \alpha_0(t)I + \alpha_1(t)A$ for some scalar functions $\alpha_0(\cdot)$ and $\alpha_1(\cdot)$. Explicitly compute these scalar functions.

Problem 3

Internal stability (4 points). Consider the system $\dot{x} = Ax$ with

1. $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
2. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Discuss the stability of the origin of each system.

Problem 4

Exponential stability and Controllability (6 points). S

Suppose that the LTI system (A, B) is completely controllable, and that there exists a symmetric, positive definite matrix P such that

$$AP + PA^T = -BB^T$$

Show that all eigenvalues of A have negative real parts.

Hint: If λ is an eigenvalue of A, and e is the corresponding left eigenvector, then $e \neq 0$ and $e^T A = \lambda e^T$. Also, $\bar{e}$ (complex conjugate) is a left eigenvalue for the eigenvalue $\bar{\lambda}$.

Problem 5

Controllability (4 points). For each of the following, provide either a proof or a counterexample:

- Suppose (A, B) is controllable. Is the system (A^2, B) controllable?
- Suppose (A^2, B) is controllable. Is the system (A, B) controllable?

Problem 6

Controllability and Observability of Augmented Systems (8 points).

Suppose that the single input single output systems $L_1 = (A_1, b_1, c_1^T)$ and $L_2 = (A_2, b_2, c_2^T)$ are each completely controllable and completely observable. Discuss the controllability and observability of the systems:

$$L_3 = \left(\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, [c_1^T, c_2^T] \right)$$

$$L_4 = \left(\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}, \begin{bmatrix} c_1^T & 0^T \\ 0^T & c_2^T \end{bmatrix} \right)$$

in the two cases:

- when A_1 and A_2 have no common eigenvalues;
- when A_1 and A_2 have at least one eigenvalue in common.

Problem 7

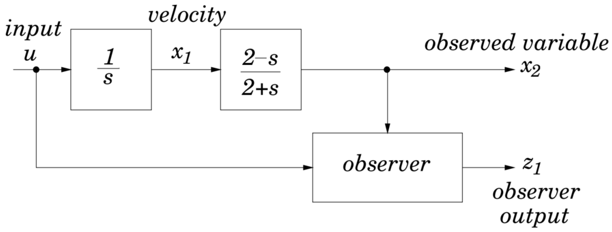
Designing B for Controllability (6 points).
Consider the following system matrix A :

$$A = \begin{bmatrix} -3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}$$

Find the minimum number of inputs needed for the system to be controllable. Find a corresponding matrix B that ensures that the system is controllable.

Problem 8

Observer design (10 points). The Figure below shows a velocity observation system where x_1 is the velocity to be observed.



An observer is to be constructed to track x_1, using u and x_2 as inputs. The variable x_2 is obtained from x_1 through a sensor having the known transfer function $\frac{2-s}{2+s}$ as shown in the Figure above.

1. **(3 points)** Derive a set of state-space equations for the system with state variables x_1 and x_2, input u and output x_2.

2. **(2 points)** Design an observer with states z_1 and z_2 to track x_1 and x_2 respectively. Choose both observer eigenvalues to be at -4. Write out the state space equations for the observer.

3. **(3 points)** Derive the combined state equation for the system plus observer. Take as state variables x_1, x_2, $e_1 = x_1 - z_1$, and $e_2 = x_2 - z_2$. Take u as input and z_1 as the output. Is this system controllable and/or observable? Give physical reasons for any states being uncontrollable or unobservable.

4. **(2 points)** What is the transfer function relating u to z_1? Explain your result.