EE650 Linear System Theory
Midterm

Your answers must be supported by analysis, proof, or counterexample. If you are asked to find the conditions under which a property holds, try to seek the most general conditions (the least restrictive conditions). There are 7 questions: approximate points for each question are indicated. The total number of possible points for the midterm is 44.

Useful Inverse Laplace Transform: $\mathcal{L}^{-1}(\frac{a}{s-b}) = ae^{bt}$

Problem 1

Vector spaces, linear independence (5 points). Suppose that the set of vectors $\{v_1, v_2 \ldots, v_n\}$ is linear dependent in V and $w \in V$. Prove that if $\{v_1 + w, v_2 + w, \ldots, v_n + w\}$ is a set of linearly dependent vectors then $w \in \text{span}\{v_1, v_2 \ldots, v_n\}$.

Problem 2

Properties of Linear maps (10 points).

1. Consider a linear map $\mathcal{A} : \mathbb{R}^4 \to \mathbb{R}^2$ such that $N(\mathcal{A}) = \{x \in \mathbb{R}^4 : x_1 = 2x_2 \text{ and } x_3 = 8x_4\}$. Is $\mathcal{A}$ subjective? Why or why not?

2. Let $\mathcal{B} : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map. Can $\mathcal{B}$ be subjective? Explain.

Problem 3

Properties of linear equations (6 points). Suppose that you are given a vector $y \in \mathbb{R}^m$ and a matrix $A \in \mathbb{R}^{m \times n}$

1. (3 points) Under which condition(s) does there exist a solution $x \in \mathbb{R}^n$ to the linear equation $y = Ax$?
2. (3 points) Under which condition(s) is this solution unique?

Problem 4

Matrix representation of a linear map (4 points).

Consider a linear map $\mathcal{A} : \mathbb{R}^2 \to \mathbb{R}^2$ and consider two bases for $\mathbb{R}^2$:

$$E = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, B = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

Suppose that:

$$\mathcal{A}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathcal{A}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$$

Calculate the matrix representation of $\mathcal{A}$ with respect to the basis B.

Problem 5

Solutions to differential equations (5 points).

Consider the differential equation $\dot{x} = f(x)$ where

$$f(x) = \begin{bmatrix} -x_1 + x_1 x_2 \\ x_2 - x_1 x_2 \end{bmatrix}$$

Derive a local Lipschiz constant for the vector field.

Problem 6

Adjoints (4 points).

Let $\mathcal{A} : U \to V$ be a linear map, with U and V Hilbert spaces. Show that $R(\mathcal{A}) \perp N(\mathcal{A}^*)$.

Problem 7

Solution to linear time-invariant system (10 points).

Consider a single-input, single-output, time invariant linear state equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; C = \begin{bmatrix} 1 & 1 \end{bmatrix};$$

and $t_0 = 0$.

1. Suppose $u(t) = 0$ for $t \geq 0$. Describe what happens to the output signal for any non-zero initial state.
2. Now suppose $u(t) = 1$ for $t \geq 1$. Describe what happens to the output signal for any non-zero initial state.